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Transfer Orbits in the Earth–Moon System Using a Regularized Model

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In a continuation of previous research where the problem was studied for the Earth–sun system, we search for transfer orbits from one body back to the same body (known in the literature as Hénon's problem) in the Earth–moon system. In particular, we are searching for orbits that can be used in three situations: 1) to transfer a spacecraft from the moon back to the moon (passing close to the Lagrangian point L_3 in most of the cases); 2) to transfer a spacecraft from the moon to the respective Lagrangian points L_3, L_4, and L_5; and 3) to transfer a spacecraft to an orbit that passes close to the moon and to the Earth several times, with the goal of building a transportation system between these two celestial bodies. The model used for the dynamics is the planar and circular restricted three-body problem. The global Lemaître regularization is used to avoid numerical problems during close approaches. An interesting result that was obtained in this research is a family of transfer orbits from the moon back to the moon that requires an impulse with magnitude lower than the escape velocity from the moon.

Introduction

TO solve the problem of finding transfer orbits in the Earth–moon system, we study several situations individually. In the first situation, attention is given to the family of transfer orbits that can transfer a spacecraft from the moon back to the moon again (passing close to the Lagrangian point L_3 in most of the cases), with minimum fuel consumption. Some of the trajectories that we found are plotted. The family of transfer orbits from the moon back to the moon that requires an impulse with magnitude lower than the escape velocity from the moon is studied and explained separately.

The problem of sending a spacecraft from the moon to the Lagrangian points L_3, L_4, and L_5 (in the Earth–moon system) is treated as a natural extension of the problem of sending a spacecraft from the moon back to the moon. Several transfer orbits are found.

In the next step, we searched for and found a trajectory that leaves the moon with a small impulse and makes several close approaches with the Earth and with the moon.

In general, the orbits found in this paper can be applied to three situations:

1) They may be used to transfer a spacecraft from the moon back to the moon with a low fuel consumption (some impulses are even lower than the escape velocity from the moon). This maneuver can be used to send a spacecraft to collect data in the Earth–moon environment from a space station fixed in the moon.

2) They may be used to transfer a spacecraft between any two points in the group formed by the moon and the Lagrangian points L_3, L_4, and L_5 (in the Earth–moon system). With those orbits we can design a mission to make a tour to the Lagrangian points for reconnaissance purposes (as suggested by Bender[1]).

3) They may be used to build a transportation system linking the moon and the Earth, using a trajectory that makes several consecutive passages close to those two bodies with no need of intermediate maneuvers.

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This research is an extension of a similar study performed for the sun–Earth system.[2,3]

Mathematical Model and Some Properties

The model used in all phases of this paper is the well-known planar circular restricted three-body problem. This model assumes that two main bodies (M_1 and M_2) are orbiting their common center of mass in circular Keplerian orbits and a third body (M_3), with negligible mass, is orbiting these two primaries. The motion of M_3 is constrained to stay in the plane of the motion of M_1 and M_2 and it is affected by both primaries, but it does not affect their motion.[4] The standard canonical system of units associated with this model (the unit of distance is the distance between M_1 and M_2 and the unit of time is chosen such that the period of the motion of M_2 around M_1 is 2π) is used. Under this model, the equations of motion are

$$\ddot{x} - 2\dot{y} = x - \frac{\partial V}{\partial x} = \frac{\partial \Omega}{\partial x} \tag{1a}$$

$$\ddot{y} + 2\dot{x} = y - \frac{\partial V}{\partial y} = \frac{\partial \Omega}{\partial y} \tag{1b}$$

where Ω is the pseudopotential function given by

$$\Omega = \tfrac{1}{2}(x^2 + y^2) + (1 - \mu)/r_1 + (\mu/r_2) \tag{2}$$

and x and y are two perpendicular axes with the origin in the center of mass of the system, with x pointing from M_1 (which has coordinates $x = -\mu$, $y = 0$) to M_2 (which has coordinates $x = 1 - \mu$, $y = 0$). Figure 1 shows the geometry of the three bodies.

One of the most important reasons why the rotating frame is more suitable to describe the motion of M_3 in the three-body problem is the existence of an invariant that is called the Jacobi integral. There are many ways to define the Jacobi integral and the reference system used to describe this problem (see Ref. 4, p. 449). In this paper the definitions used by Broucke[5] are followed. In this version, the Jacobi integral is given by

$$J = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) - \Omega(x, y) = \text{const} \tag{3}$$

The equations of motion given by Eqs. (1) are correct, but they are not suitable for numerical integration in trajectories passing near one of the primaries. The reason is that the positions of both primaries

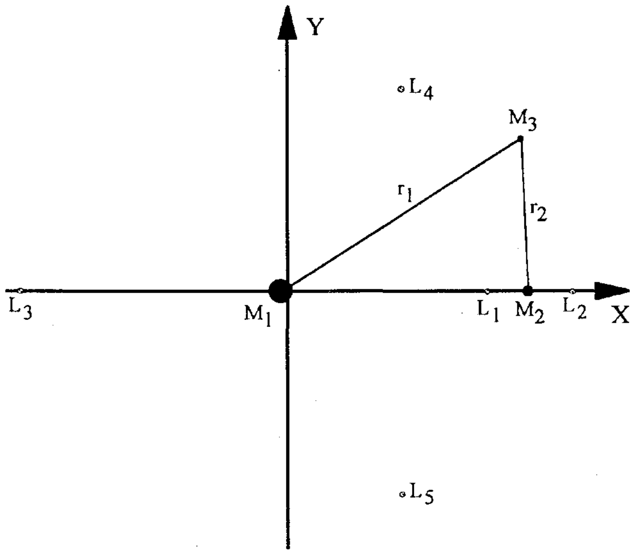

Fig. 1 Geometry of the problem of three bodies.

are singularities in the potential (since r_1 or r_2 goes to zero, or near zero), and the accuracy of the numerical integration is affected every time this situation occurs.

Lemaître Regularization

The solution for this problem is to use regularization that consists of a substitution of the variables for position (x, y) and time (t) by another set of variables $(\omega_1, \omega_2, \tau)$, such that the singularities are eliminated in these new variables. Several transformations with this goal are available in the literature (see Ref. 4, Chap. 3), such as those of Thiele-Burrau, Lemaître, and Birkhoff. They are called global regularization, to emphasize that both singularities are eliminated at the same time. The case where only one singularity is eliminated at a time is called local regularization. For the present research Lemaître's regularization is used. To perform the required transformation, it is necessary first to define a new complex variable $q = q_1 + i \times q_2 (i = \sqrt{-1}$ is the imaginary unit), with q_1 and q_2 given by

$$q_1 = x + \tfrac{1}{2} - \mu \tag{4}$$

$$q_2 = y \tag{5}$$

Now, in terms of q, the transformation involved in Lemaître regularization is given by

$$q = f(\omega) = \tfrac{1}{4}[\omega^2 + (1/\omega^2)] \tag{6}$$

for the old variables for position (x, y) and

$$\frac{\partial t}{\partial \tau} = |f'(\omega)|^2 = \frac{|\omega^4 - 1|^2}{4|\omega|^6} \tag{7}$$

where $f'(\omega)$ denotes $\partial f/\partial \omega$ for the time.

In these new variables the equation of motion of the system is

$$\omega'' + 2i|f'(\omega)|^2\omega' = \mathrm{grad}_\omega \Omega^* \tag{8}$$

where $\omega = \omega_1 + i \times \omega_2$ is the new complex variable for position, ω' and ω'' denote first and second derivatives of ω with respect to the regularized time τ, $\mathrm{grad}_\omega\Omega^*$ represents $(\partial\Omega^*/\partial\omega_1) + i(\partial\Omega^*/\partial\omega_2)$ and Ω^* is the transformed pseudopotential given by

$$\Omega^* = [\Omega - (C/2)]|f'(\omega)|^2 \tag{9}$$

where $C = \mu(1 - \mu) - 2J$.

Equation (8) in complex variables can be separated in two second-order equations in the real variables ω_1 and ω_2 and organized in the standard first-order form that is more suitable for numerical integration. The final form, after defining the regularized velocity components ω_3 and ω_4 as $\omega_1' = \omega_3$ and $\omega_2' = \omega_4$, is

$$\omega_1' = \omega_3 \tag{10a}$$

$$\omega_2' = \omega_4 \tag{10b}$$

$$\omega_3' = 2\omega_4|f'(\omega)|^2 + \frac{\partial\Omega^*}{\partial\omega_1} \tag{10c}$$

$$\omega_4' = -2\omega_3|f'(\omega)|^2 + \frac{\partial\Omega^*}{\partial\omega_2} \tag{10d}$$

Another set of equations necessary for this research is the one that relates velocity components from one set of variables to another. They are

$$\dot{q}_1 = \frac{f'(\omega)}{|f'(\omega)|^2}\omega_3 \tag{11a}$$

$$\dot{q}_2 = \frac{f'(\omega)}{|f'(\omega)|^2}\omega_4 \tag{11b}$$

Another important property needed in this paper is the mirror image theorem.[6] It is an important and useful property of the planar circular restricted three-body problem. It says that in the rotating coordinate system, for each trajectory defined by $x(t), y(t), \dot{x}(t), \dot{y}(t)$ that is found, there is a symmetric (in relation to the x axis) trajectory defined by $x(-t), -y(-t), -\dot{x}(-t), -\dot{y}(-t)$.

Lagrangian Points and Their Applications

The well-known Lagrangian points that appear in the planar restricted three-body problem (see Fig. 1) are very important for astronautical applications.[7] They are five points of equilibrium in the equations of motion, which means that a particle located at one of those points with zero velocity will remain there indefinitely. The collinear points (L_1, L_2, and L_3) are always unstable and the triangular points (L_4 and L_5) are stable in the present case studied (Earth–moon system).

They are all very good points to locate a space station, since they require a small amount of ΔV (and fuel) for station keeping. The triangular points are especially good for this purpose, since they are stable equilibrium points.

The cislunar point L_1 is an important option as a node to explore the moon[8] because it can be used as a parking orbit to store spacecraft, fuel, and supplies that are needed for the return journey to the Earth but are not needed at the moon.

The Lagrangian point L_2 is very useful as a place to keep a relay satellite[7] since a satellite in an orbit around this point could see the far side of the moon and the Earth at the same time indefinitely.

In the nomenclature used in this paper, L_3 is the collinear Lagrangian point that exists on the opposite side of the Earth (when compared to the position of the moon). It is located about 390,000 km from the Earth, which means that it is almost at the same distance as the moon, but in the opposite direction.

One of the triangular Lagrangian points is called L_4. Its location is the third vertex of the equilateral triangle formed with the Earth and the moon, in the semiplane of positive y. In the present case (Earth–moon system) it is a stable equilibrium point. It is a very important point, because it is an excellent location for a space station. Its stability property makes the fuel required for station keeping almost zero.

The other triangular Lagrangian point is L_5. Its location is the point symmetric to L_4 (in relation to the horizontal axis shown in Fig. 1), the third vertex of the equilateral triangle formed with the Earth and the moon, in the semiplane of negative y. It is also stable and a very important point, for the same reasons that L_4 is an important point.

Results

The theory developed in our previous research[2,3,9] to solve the problem of transferring a spacecraft from one body back to the same body can be used here to solve Hénon's problem[10] in the case $\mu \neq 0$

for the Earth–moon system. The approach used here is to solve the two-point boundary value problem with the following input data: 1) the initial position of the spacecraft is the position of the moon at the time that the spacecraft departs from the moon and 2) the final position of the spacecraft is the position of the moon at the time that the spacecraft arrives at the moon.

The solution of the problem (output of the two-point boundary value problem) is the desired transfer orbit (in the restricted three-body context). The scheme looks very simple, but it is not easy to implement. The difficulty arises from that to get convergence in the solution of the two-point boundary value problem involved an accurate first guess is required for each transfer orbit considered. The first good first guess available is the solution of the related two-body problem (same initial and final position, but with $\mu = 0$, the mass of the moon is neglected, using two-body celestial mechanics equations). Using this method to start the process and a trial-and-error technique to get the final results, we were able to find several interesting and useful trajectories. They are shown, in families, in the next sections.

Transfer Orbits with ΔV Near the Escape Velocity

In this section, the theory explained in the first sections of this paper is used to find transfer orbits from the moon back to the moon with ΔV near the escape velocity.

An important characteristic of Hénon's problem,[10] using the two-body celestial mechanics as a model, is the family of transfer orbits with near zero ΔV to transfer a spacecraft from the moon back to the moon again.[3,9] These orbits, which exist in the two-body problem model (case $\mu = 0$ of the restricted three body problem), are important enough to deserve study in the more realistic case $\mu \neq 0$. In the present paper, this research is performed for the Earth–moon system. The two-body solution is used as the first guess and a trial-and-error technique (in the initial velocity) is used to find the desired trajectory. Figure 2 shows three of those trajectories, as seen in the rotating frame. They are trajectories that leave the moon with the application of a single impulse, go close to the Lagrangian point L_3 (in the opposite side of the moon), and then return to the moon. There is no need of any impulse other than the initial one. They are thrust-free trajectories. They are similar to Hénon's periodic consecutive orbits, but they are in the restricted three-body problem. Note that the ΔV for escape velocity from the moon at the distance that we used (0.0045 canonical units = 1730 km) is 2.321739099 canonical units (the absolute minimal for any transfer from the surface of a celestial body), which means that the ΔV found in those transfer orbits (2.333604379, 2.357820012, and 2.329100000 canonical units) are only a little bit above the escape velocity (0.01186528, 0.036080913, and 0.007360901 canonical units, respectively), and there is not much improvement left to be made, as far as fuel savings are concerned. Table 1 shows the initial conditions for those trajectories in canonical units and Table 2 shows the same results expressed in kilometers and kilometer per second. The value 0.0045 is used very often in this paper because it corresponds to an altitude of 100 km above the surface of the moon, which is a value considered by many mission analysts in the literature. We did not use any specific constraint for the terminal points. The only requirement is that the spacecraft returns close to the moon.

Transfer with ΔV Under the Escape Velocity

During the study of the trajectories with ΔV near the escape velocity, a new type of transfer was found. This new type requires an impulse with a magnitude lower than the escape velocity. This possibility is opened by the restricted three-body problem model. It uses the perturbation of the third body (the Earth in this case) to help the second escape from the moon (the first body), and it decreases the ΔV required. Remember that the escape velocity is defined as the velocity required to escape one celestial body considering the system governed by two-body celestial mechanics. Figure 3 shows three of those transfers. Tables 1 and 2 show the initial conditions for those trajectories. The trajectory T4 was obtained by backward integration.

Table 1 Initial conditions for the trajectories shown (canonical units)

Traj.	x	y	$\dot{x}$	$\dot{y}$	Vesc	$\Delta V - V$esc
T1	0.987871437	−0.004521000	0.000000000	−2.333600000	2.316340566	0.017263813
T2	0.987871437	−0.004500000	0.450000000	−2.313600000	2.321739099	0.036080913
T3	0.992371437	0.000000000	0.000000000	2.333600000	2.321739099	0.007360901
T4	0.987871437	−0.004786681	2.220000000	0.000000000	2.251139608	−0.031139608
T5	0.992371437	0.000000000	0.000000000	2.300000000	2.321739099	−0.026239098
T6	0.987871437	0.000000000	0.000000000	2.300000000	2.321739099	−0.017239098
T7	0.987871437	−0.004500000	0.000000000	−3.013600000	2.321739099	0.691864261
T8	0.987871437	−0.004500000	0.100000000	−3.063600000	2.321739099	0.743642641
T9	0.987871437	−0.004500000	0.000000000	−3.053600000	2.321739099	0.731864217
T10	0.987871437	−0.004500000	−0.100000000	−3.063600000	2.321739099	0.743349025
T11	0.987871437	−0.004500000	2.500000000	0.000000000	2.321739099	0.182760901

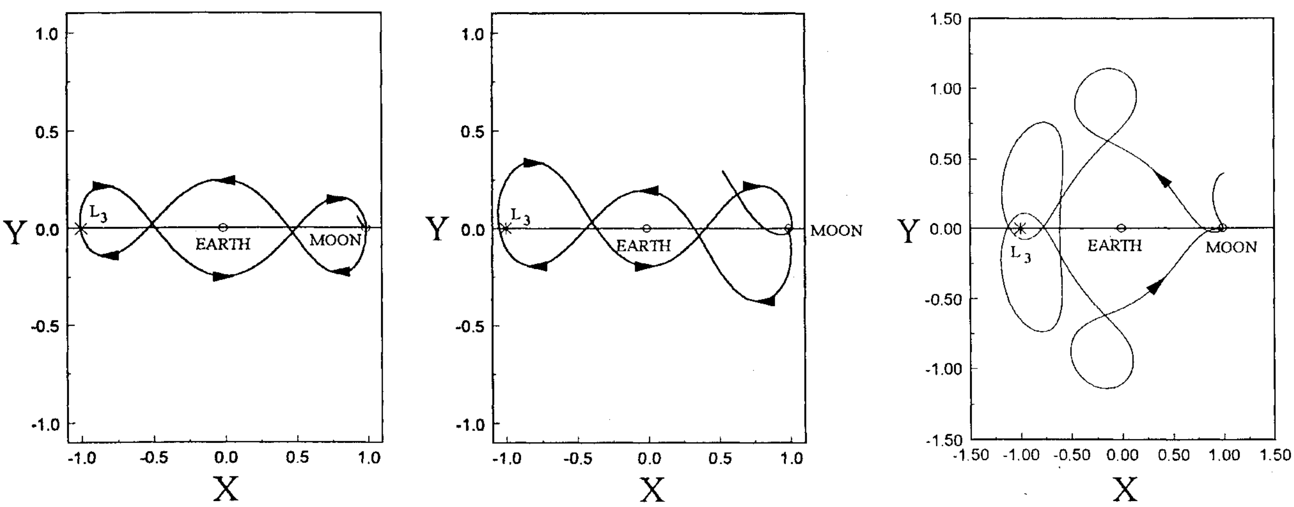

Fig. 2 Transfers from the moon back to the moon (T1 , T2, and T3).

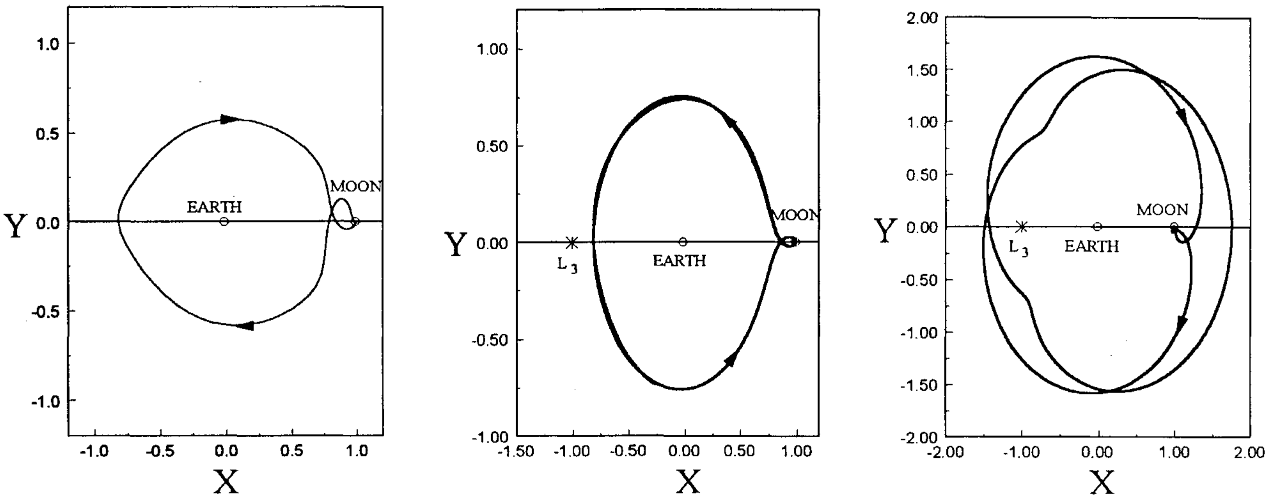

Fig. 3 Transfer from the moon back to the moon with $\Delta V - V\text{esc} < 0$ (T4, T5, and T6).

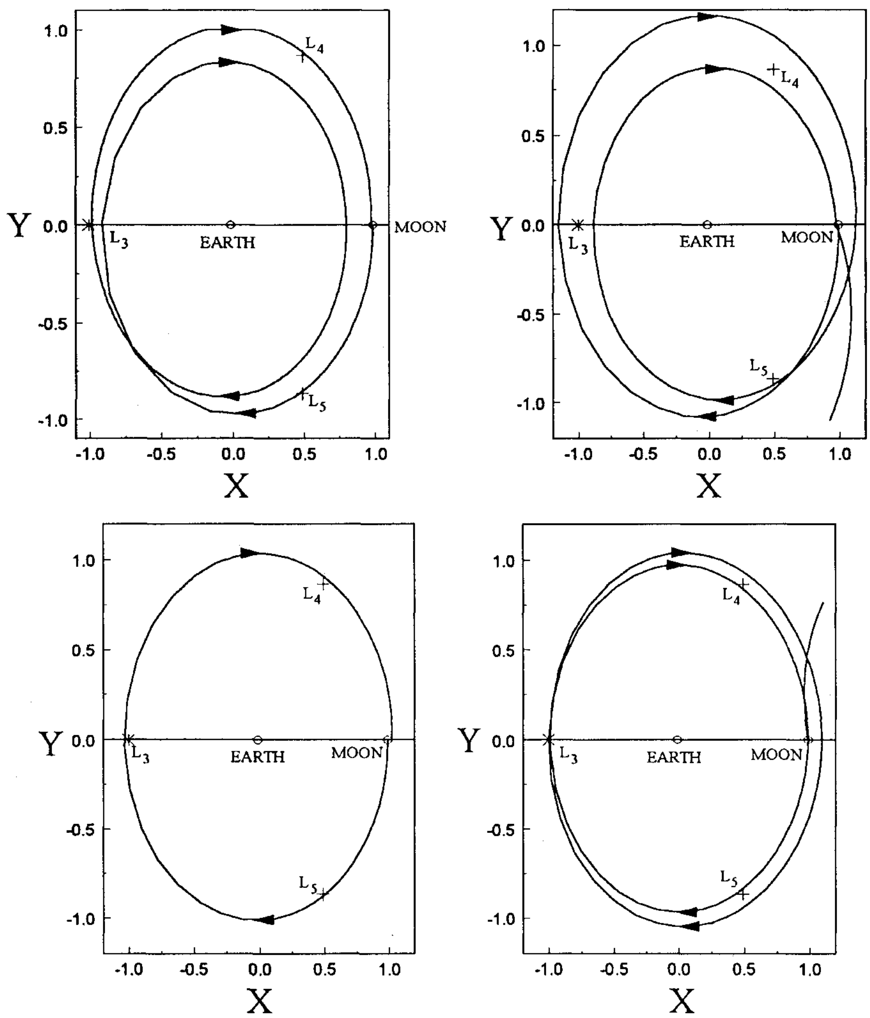

Fig. 4 Transfers from the moon to the Lagrangian points (T7, T8, T9, and T10).

Transfers Between the Moon and the Lagrangian Points

In this section, we study transfers between the moon and the Lagrangian points in the Earth–moon system. Figure 4 shows some of the transfers from the moon to the Lagrangian points and back to the moon. We show four options for this mission, where we can choose the best one according to the mission requirements, such as time available for the duration of the mission, approximation desired to each of the Lagrangian points visited, etc. Three of those trajectories pass twice by the Lagrangian points visited. Tables 1 and 2 show the initial conditions for those trajectories. Those options do not have a ΔV near the escape velocity, but they do have some other attractions, such as a relatively small time of flight, two passages by each point with no more maneuvers required, etc.

There are some other excellent trajectories for transfers from the moon to L_4 and L_5 using periodic orbits around the Lagrangian points L_1 and L_2. Those orbits can build a permanent transportation system linking all of the points (moon, L_4, L_5) with no need of more impulses. They are shown by Broucke.[5]

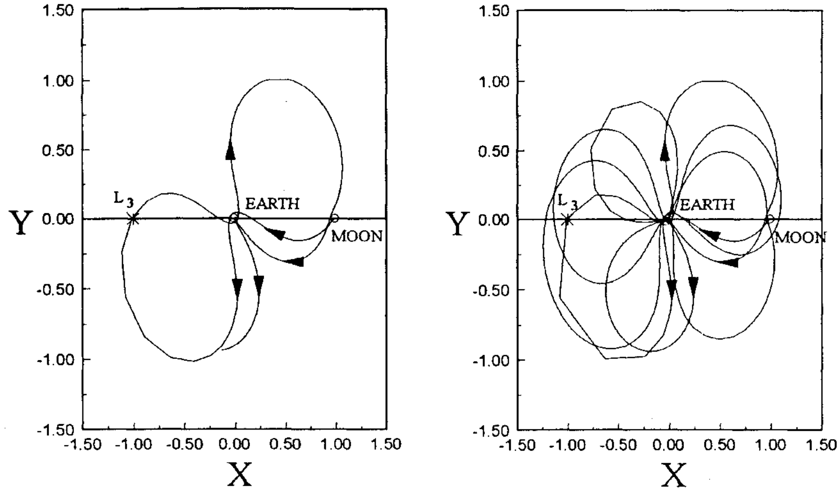

Fig. 5 Trajectory linking the moon and the Earth (T11).

Table 2 Initial conditions for the trajectories shown (km and km/s)

Traj.	x	y	$\dot{x}$	$\dot{y}$	Vesc	$\Delta V - V$esc
T1	379737.78	−1737.87	0.0000	−2.3873	2.3696	0.0177
T2	379737.78	−1729.80	0.4604	−2.3668	2.3751	0.0369
T3	381467.58	0.00	0.0000	2.3873	2.3751	0.0075
T4	379737.78	−1840.00	2.2711	0.0000	2.3029	−0.0319
T5	381467.58	0.00	0.0000	2.3529	2.3751	−0.0268
T6	379737.78	0.00	0.0000	2.3529	2.3751	−0.0176
T7	379737.78	−1729.80	0.0000	−3.0829	2.3751	0.7078
T8	379737.78	−1729.80	0.1023	−3.1341	2.3751	0.7607
T9	379737.78	−1729.80	0.0000	−3.1238	2.3751	0.7487
T10	379737.78	−1729.80	−0.1023	−3.1341	2.3751	0.7604
T11	379737.78	−1729.80	2.5575	0.0000	2.3751	0.1870

Transfers Between the Moon and the Earth

The next topic treated in this paper is the study of trajectories that can connect the moon and the Earth. A very good representative of this type of trajectory is T11, shown in Fig. 5 (the first revolutions are detailed at the left side and more revolutions at the right side). The trajectory leaves the moon and several times passes very close to the Earth and the moon again. It also passes by the Lagrangian point L_3. There are several practical applications for an orbit of this type: for example, to collect scientific data in space and return it to the Earth during the close approaches and to build a transportation system linking the Earth and the moon.

Conclusions

From the analyses of the results shown in this paper we can draw the following conclusions: 1) There are trajectories with ΔV near the escape velocity to move a spacecraft from the moon back to the moon in the Earth–moon system, using the restricted three-body problem with Lemaître regularization as a model. 2) There is a new type of trajectory for this transfer that requires a ΔV under the escape velocity. 3) There are trajectories connecting the moon and the Lagrangian points L_3, L_4, and L_5. 4) There are trajectories that make consecutive close approaches with the Earth and the moon. Those orbits are shown in this paper, and they can be used in three situations: to transfer a spacecraft from the moon back to the moon; to transfer a spacecraft from the moon to the respective Lagrangian points L_3, L_4, and L_5; and to transfer a spacecraft to an orbit that passes close to the moon and to the Earth several times, with the goal of building a transportation system between these two celestial bodies.

Acknowledgments

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